THE

ANALYST;

OR, A

DISCOURSE

Addreſſed to an
Infidel MATHEMATICIAN.

WHEREIN

It is examined whether the Object, Principles, and Inferences of the modern Analyſis are more diſtinctly conceived, or more evidently deduced, than Religious Myſteries and Points of Faith.

By the Author of *The Minute Philoſopher.*

Firſt caſt out the beam out of thine own Eye; and then ſhalt thou ſee clearly to caſt out the mote out of thy brother's Eye.
S. Matt. c. vii. 5.

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THE

ANALYST.

I. THOUGH I am a Stranger to your Perſon, yet I am not, Sir, a Stranger to the Reputation you have acquired, in that Branch of Learning which hath been your peculiar Study; nor to the Authority that you therefore aſſume in things foreign to your Profeſſion, nor to the Abuſe that you, and too many more of the like Character, are known to make of ſuch undue Authority, to the miſleading of unwary Perſons in matters of the higheſt Concernment, and whereof your mathematical Knowledge can by no means qualify you to be a competent Judge. Equity indeed and good Senſe would incline one to diſregard the Judgment of Men, in Points which they have not conſidered or examined. But ſeveral who make the loudeſt Claim to thoſe Qualities, do, nevertheleſs, the very thing they would ſeem to deſpiſe, clothing themſelves in the Livery of other Men's Opinions, and putting on a general Deference for the Judgment of you, Gentlemen, who are preſumed to be of all Men the greateſt Maſters of Reaſon, to be moſt converſant about diſtinct Ideas, and never to take things upon truſt, but always clearly to ſee your way, as Men whoſe conſtant Employment is the deducing Truth by the juſteſt Inference from the moſt evident Principles. With this Bias on their Minds, they ſubmit to your Deciſions where you have no right to decide. And that this is one ſhort way of making Infidels I am credibly informed.

II. Whereas then it is ſuppoſed, that you apprehend more diſtinctly, conſider more cloſely, infer more juſtly, conclude more accurately than other Men, and that you are therefore leſs religious becauſe more judicious, I ſhall claim the Privilege of a Free-Thinker; and take the Liberty to inquire into the Object, Principles, and Method of Demonſtration admitted by the Mathematicians of the preſent Age, with the ſame Freedom that you preſume to treat the Principles and Myſteries of Religion; to the end, that all Men may ſee what Right you have to lead, or what Encouragement others have to follow you. It hath been an old Remark that Geometry is an excellent Logic. And it muſt be owned, that when the Definitions are clear; when the Poſtulata cannot be refuſed, nor the Axioms denied; when from the diſtinct Contemplation and Compariſon of Figures, their Properties are derived, by a perpetual well-connected Chain of Conſequences, the Objects being ſtill kept in view, and the Attention ever fixed upon them; there is acquired an Habit of Reaſoning, cloſe and exact and methodical: which Habit ſtrengthens and ſharpens the Mind, and being transferred to other Subjects, is of general Uſe in the Inquiry after Truth. But how far this is the caſe of our Geometrical Analyſts, it may be worth while to conſider.

III. The Method of Fluxions is the general Key, by help whereof the modern Mathematicians unlock the Secrets of Geometry, and conſequently of Nature. And as it is that which hath enabled them ſo remarkably to outgo the Antients in diſcovering Theorems and ſolving Problems, the Exerciſe and Application thereof is become the main, if not ſole, Employment of all thoſe who in this Age paſs for profound Geometers. But whether this Method be clear or obſcure, conſiſtent or repugnant, demonſtrative or precarious, as I ſhall inquire with the utmoſt Impartiality, ſo

I ſubmit my Inquiry to your own Judgment, and that of every candid Reader. Lines are ſuppoſed to be generated * by the Motion of Points, Planes by the motion of Lines, and Solids by the motion of Planes. And whereas Quantities generated in equal times are greater or leſſer, according to the greater or leſſer Velocity, wherewith they increaſe and are generated, a Method hath been found to determine Quantities from the Velocities of their generating Motions. And ſuch Velocities are called Fluxions: and the Quantities generated are called flowing Quantities. Theſe Fluxions are ſaid to be nearly as the Increments of the flowing Quantities, generated in the leaſt equal Particles of Time; and to be accurately in the firſt Proportion of the naſcent, or in the laſt of the evaneſcent Increments. Sometimes, inſtead of Velocities, the momentaneous Increments or Decrements of undetermined flowing Quantities are conſidered, under the Appellation of Moments.

IV. By Moments we are not to underſtand finite Particles. Theſe are ſaid not to be Moments, but Quantities generated from Moments, which laſt are only the naſcent Principles of finite Quantities. It is ſaid, that the minuteſt Errors are not to be neglected in Mathematics: that the Fluxions are Celeri-

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* Introd. ad Quadraturam Curvarum.

ties, not proportional to the finite Increments though ever ſo ſmall; but only to the Moments or naſcent Increments, whereof the Proportion alone, and not the Magnitude, is conſidered. And of the aforeſaid Fluxions there be other Fluxions, which Fluxions of Fluxions are called ſecond Fluxions. And the Fluxions of theſe ſecond Fluxions are called third Fluxions: and ſo on, fourth, fifth, ſixth, *&c. ad infinitum.* Now as our Senſe is ſtrained and puzzled with the Perception of Objects extremely minute, even ſo the Imagination, which Faculty derives from Senſe, is very much ſtrained and puzzled to frame clear Ideas of the leaſt Particles of Time, or the leaſt Increments generated therein: and much more ſo to comprehend the Moments, or thoſe Increments of the flowing Quantities in *ſtatu naſcenti*, in their very firſt Origin or Beginning to exiſt, before they become finite Particles. And it ſeems ſtill more difficult, to conceive the abſtracted Velocities of ſuch naſcent imperfect Entities. But the Velocities of the Velocities, the ſecond, third, fourth and fifth Velocities, *&c.* exceed, if I miſtake not, all Human Underſtanding. The further the Mind analyſeth and purſueth theſe fugitive Ideas, the more it is loſt and bewildered; the Objects, at firſt fleeting and minute, ſoon vaniſhing out of ſight. Certainly in any Senſe a ſecond or third Fluxion ſeems an obſcure Myſtery. The incipient Celerity of an incipient

Celerity, the naſcent Augment of a naſcent Augment, *i. e.* of a thing which hath no Magnitude: Take it in which Light you pleaſe, the clear Conception of it will, if I miſtake not, be found impoſſible, whether it be ſo or no I appeal to the trial of every thinking Reader. And if a ſecond Fluxion be inconceivable, what are we to think of third, fourth, fifth Fluxions, and ſo onward without End?

V. The foreign Mathematicians are ſuppoſed by ſome, even of our own, to proceed in a manner, leſs accurate perhaps and geometrical, yet more intelligible. Inſtead of flowing Quantities and their Fluxions, they conſider the variable finite Quantities, as increaſing or diminiſhing by the continual Addition or Subduction of infinitely ſmall Quantities. Inſtead of the Velocities wherewith Increments are generated, they conſider the Increments or Decrements themſelves, which they call Differences, and which are ſuppoſed to be infinitely ſmall. The Difference of a Line is an infinitely little Line; of a Plane an infinitely little Plane. They ſuppoſe finite Quantities to conſiſt of Parts infinitely little, and Curves to be Polygons, whereof the Sides are infinitely little, which by the Angles they make one with another determine the Curvity of the Line. Now to conceive a Quantity infinitely ſmall, that is, infinitely leſs than any ſenſible or imaginable Quantity, or any the leaſt finite Magnitude, is, I confeſs, above my Capacity. But to conceive a Part of ſuch infinitely ſmall Quantity, that ſhall be ſtill infinitely leſs than it, and conſequently though multiply'd infinitely ſhall never equal the minuteſt finite Quantity, is, I ſuſpect, an infinite Difficulty to any Man whatſoever; and will be allowed ſuch by thoſe who candidly ſay what they think; provided they really think and reflect, and do not take things upon truſt.

VI. And yet in the *calculus differentialis*, which Method ſerves to all the ſame Intents and Ends with that of Fluxions, our modern Analyſts are not content to conſider only the Differences of finite Quantities: they alſo conſider the Differences of thoſe Differences, and the Differences of the Differences of the firſt Differences. And ſo on *ad infinitum*. That is, they conſider Quantities infinitely leſs than the leaſt diſcernible Quantity; and others infinitely leſs than thoſe infinitely ſmall ones; and ſtill others infinitely leſs than the preceding Infiniteſimals, and ſo on without End or Limit. Inſomuch that we are to admit an infinite Succeſſion of Infiniteſimals, each infinitely leſs than the foregoing, and infinitely greater than the following. As there are firſt, ſecond, third, fourth, fifth, &c. Fluxions, ſo there are Differences, firſt, ſecond, third, fourth &c. in an infinite Progreſſion towards nothing, which you ſtill approach and never arrive at.

And (which is moſt ſtrange) although you ſhould take a Million of Millions of theſe Infiniteſimal, each whereof is ſuppoſed infinitely greater than ſome other real Magnitude, and add them to the leaſt given Quantity, it ſhall be never the bigger. For this is one of the modeſt *poſtulata* of our modern Mathematicians, and is a Corner-ſtone or Ground-work of their Speculations.

VII. All theſe Points, I ſay, are ſuppoſed and believed by certain rigorous Exactors of Evidence in Religion, Men who pretend to believe no further than they can ſee. That Men, who have been converſant only about clear Points, ſhould with Difficulty admit obſcure ones might not ſeem altogether unaccountable. But he who can digeſt a ſecond or third Fluxion, a ſecond or third Difference, need not, methinks, be ſqueamiſh about any Point in Divinity. There is a natural Preſumption that Mens Faculties are made alike. It is on this Suppoſition that they attempt to argue and convince one another. What, therefore, ſhall appear evidently impoſſible and repugnant to one, may be preſumed the ſame to another. But with what Appearance of Reaſon ſhall any Man preſume to ſay, that Myſteries may not be Objects of Faith, at the ſame time that he himſelf admits ſuch obſcure Myſteries to be the Object of Science?

VIII. It muſt indeed be acknowledged, the modern Mathematicians do not conſider theſe Points as Myſteries, but as clearly conceived and maſtered by their comprehenſive Minds. They ſcruple not to ſay, that by the help of theſe new Analytics they can penetrate into Infinity it ſelf: That they can even extend their Views beyond Infinity: that their Art comprehends not only Infinite, but Infinite of Infinite (as they expreſs it) or an Infinity of Infinites. But, notwithſtanding all theſe Aſſertions and Pretenſions, it may be juſtly queſtioned whether, as other Men in other Inquiries are often deceived by Words or Terms, ſo they likewiſe are not wonderfully deceived and deluded by their own peculiar Signs, Symbols, or Species. Nothing is eaſier than to deviſe Expreſſions or Notations for Fluxions and Infiniteſimals of the firſt, ſecond, third, fourth and ſubſequent Orders, proceeding in the ſame regular Form without End or Limit $\dot{x}$. $\ddot{x}$. $\dddot{x}$. $\ddddot{x}$. *&c.* or *dx. ddx. dddx. ddddx, &c.* Theſe Expreſſions indeed are clear and diſtinct, and the Mind finds no Difficulty in conceiving them to be continued beyond any aſſignable Bounds. But if we remove the Veil and look underneath, if laying aſide the Expreſſions we ſet our ſelves attentively to conſider the things themſelves, which are ſuppoſed to be expreſſed or marked thereby, we ſhall diſcover much Emptineſs, Darkneſs, and Confuſion; nay, if

I miſtake not, direct Impoſſibilities and Contradictions. Whether this be the Caſe or no, every thinking Reader is intreated to examine and judge for himſelf.

IX. Having conſidered the Object, I proceed to conſider the Principles of this new Analyſis by Momentums, Fluxions, or Infiniteſimals; wherein if it ſhall appear that your capital Points, upon which the reſt are ſuppoſed to depend, include Error and falſe Reaſoning; it will then follow that you, who are at a loſs to conduct your ſelves, cannot with any Decency ſet up for Guides to other Men. The main Point in the Method of Fluxions is to obtain the Fluxion or Momentum of the Rectangle or Product of two indeterminate Quantities. Inaſmuch as from thence are derived Rules for obtaining the Fluxions of all other Products and Powers; be the Coefficients or the Indexes what they will, Integers or Fractions, rational or ſurd. Now this fundamental Point one would think ſhould be very clearly made out, conſidering how much is built upon it, and that its Influence extends throughout the whole Analyſis. But let the Reader judge. This is given for Demonſtration. * Suppoſe the Product or Rectangle AB increaſed by continual Motion: and that the momentaneous Increments of the Sides A and B are a and b. When the Sides A and B

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* Naturalis Philoſophiæ principia mathematica, l. 2. lem. 2.

were deficient, or leſſer by one half of their Moments, the Rectangle was $\overline{A-\frac{1}{2}a}\times\overline{B-\frac{1}{2}b}$ i. e. $AB-\frac{1}{2}aB-\frac{1}{2}bA+\frac{1}{4}ab$. And as ſoon as the Sides A and B are increaſed by the other two halves of their Moments, the Rectangle becomes $\overline{A+\frac{1}{2}a}\times\overline{B+\frac{1}{2}b}$ or $AB+\frac{1}{2}aB+\frac{1}{2}bA+\frac{1}{4}ab$. From the latter Rectangle ſubduct the former, and the remaining difference will be $aB+bA$. Therefore the Increment of the Rectangle generated by the intire Increments a and b is $aB+bA$. *Q. E. D.* But it is plain that the direct and true Method to obtain the Moment or Increment of the Rectangle AB, is to take the Sides as increaſed by their whole Increments, and ſo multiply them together, $A+a$ by $B+b$, the Product whereof $AB+aB+bA+ab$ is the augmented Rectangle; whence if we ſubduct AB, the Remainder $aB+bA+ab$ will be the true Increment of the Rectangle, exceeding that which was obtained by the former illegitimate and indirect Method by the Quantity ab. And this holds univerſally by the Quantities a and b be what they will, big or little, finite or infiniteſimal, Increments, Moments, or Velocities. Nor will it avail to ſay that ab is a Quantity exceeding ſmall: Since we are told that *in rebus mathematicis errores quàm minimi non ſunt contemnendi*

X. * Such Reaſoning as this for Demonſtration, nothing but the Obſcurity of the Subject could have encouraged or induced the great Author of the Fluxionary Method to put upon his Followers, and nothing but an implicit Deference to Authority could move them to admit. The Caſe indeed is difficult. There can be nothing done till you have got rid of the Quantity *a b*. In order to this the Notion of Fluxions is ſhifted: It is placed in various Lights: Points which ſhould be clear as firſt Principles are puzzled; and Terms which ſhould be ſteadily uſed are ambiguous. But notwithſtanding all this Addreſs and Skill the point of getting rid of *a b* cannot be obtained by legitimate Reaſoning. If a Man by Methods, not geometrical or demonſtrative, ſhall have ſatisfied himſelf of the Uſefulneſs of certain Rules; which he afterwards ſhall propoſe to his Diſciples for undoubted Truths; which he undertakes to demonſtrate in a ſubtil Manner, and by the Help of nice and intricate Notions; it is not hard to conceive that ſuch his Diſciples may, to ſave themſelves the Trouble of Thinking, be inclined to confound the Uſefulneſs of a Rule with the Certainty of a Truth, and accept the one for the other; eſpecially if they are Men accuſtomed rather to compute than to think; earneſt rather to go on faſt and far, than ſollicitous to ſet out warily and ſee their Way diſtinctly.

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* Introd. ad Quadraturam Curvarum.

XI. The Points or meer Limits of naſcent Lines are undoubtedly equal, as having no more magnitude one than another, a Limit as ſuch being no Quantity. If by a Momentum you mean more than the initial Limit, it muſt be either a finite Quantity or an Infiniteſimal. But all finite Quantities are expreſly excluded from the Notion of a Momentum. Therefore the Momentum muſt be an Infiniteſimal. And indeed, though much Artifice hath been employ'd to eſcape or avoid the admiſſion of Quantities infinitely ſmall, yet it ſeems ineffectual. For ought I ſee, you can admit no Quantity as a Medium between a finite Quantity and nothing, without admitting Infiniteſimals. An Increment generated in a finite Particle of Time, is it ſelf a finite Particle; and cannot therefore be a Momentum. You muſt therefore take an Infiniteſimal Part of Time wherein to generate your Momentum. It is ſaid, the Magnitude of Moments is not conſidered: And yet theſe ſame Moments are ſuppoſed to be divided into Parts. This is not eaſy to conceive, nor more than it is why we ſhould take Quantities leſs than *A* and *B* in order to obtain the Increment of *A B*, of which proceeding it muſt be owned the final Cauſe or Motive is obvious; but it is not ſo obvious or eaſy to explain a juſt and legitimate Reaſon for it, or ſhew it to be Geometrical.

XII. From the foregoing Principle ſo demonſtrated, the general Rule for finding the Fluxion of any Power of a flowing Quantity is derived *. But, as there ſeems to have been ſome inward Scruple or Conſciouſneſs of defect in the foregoing Demonſtration, and as this finding the Fluxion of a given Power is a Point of primary Importance, it hath therefore been judg'd proper to demonſtrate the ſame in a different Manner independent of the foregoing Demonſtration. But whether this other Method be more legitimate and concluſive than the former, I proceed now to examine; and in order thereto ſhall premiſe the following Lemma. “If with a View to demonſtrate any Propoſition, a certain Point is ſuppoſed, by virtue of which certain other Points are attained; and ſuch ſuppoſed Point be it ſelf afterwards deſtroyed or rejected by a contrary Suppoſition; in that caſe, all the other Points, attained thereby and conſequent thereupon, muſt alſo be deſtroyed and rejected, ſo as from thence forward to be no more ſuppoſed or applied in the Demonſtration.” This is ſo plain as to need no Proof.

XIII. Now the other Method of obtaining a Rule to find the Fluxion of any Power is as follows. Let the Quantity x flow uniformly, and be it propoſed to find the Fluxion of x^n. In the ſame time that x by flowing becomes $x+o$, the Power x^n becomes $\overline{x+o}|^{n}$, i. e. by the Method of infinite Series $x^n + nox^{n-1} + \frac{nn-n}{2} oox^{n-2} + \&c.$ and the Increments o and $nox^{n-1} + \frac{nn-n}{2} oox^{n-2} + \&c.$ are one to another as 1 to $nx^{n-1} + \frac{nn-n}{2} ox^{n-2} + \&c.$ Let now the Increments vaniſh, and their laſt Proportion will be 1 to nx^{n-1}. But it ſhould ſeem that this reaſoning is not fair or concluſive. For when it is ſaid, let the Increments vaniſh, *i. e.* let the Increments be nothing, or let there be no Increments, the former Suppoſition that the Increments were ſomething, or that there were Increments, is deſtroyed, and yet a Conſequence of that Suppoſition, *i. e.* an Expreſſion got by virtue thereof, is retained. Which, by the foregoing Lemma, is a falſe way of reaſoning. Certainly when we ſuppoſe the Increments to vaniſh, we muſt ſuppoſe their Proportions, their Expreſſions, and every thing elſe derived from the Suppoſition of their Exiſtence to vaniſh with them.

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* Philoſophiæ naturalis principia Mathematica, lib. 2. lem. 2.

XIV. To make this Point plainer, I ſhall unfold the reaſoning, and propoſe it in a fuller light to your View. It amounts therefore to this, or may in other Words be thus expreſſed. I ſuppoſe that the Quantity x flows, and by flowing is increaſed, and its Increment I call o, ſo that by flowing it becomes $x+o$. And as x increaſeth, it follows that every Power of x

is likewiſe increaſed in a due Proportion. Therefore as x becomes $x+o$, x^n will become $\overline{x+o}|^n$: that is, according to the Method of infinite Series, $x^n + nox^{n-1} + \frac{nn-n}{2}oox^{n-2} +$ &c. And if from the two augmented Quantities we ſubduct the Root and the Power reſpectively, we ſhall have remaining the two Increments, to wit, o and $nox^{n-1} + \frac{nn-n}{2}oox^{n-2} +$ &c. which Increments, being both divided by the common Diviſor o, yield the Quotients 1 and $nx^{n-1} + \frac{nn-n}{2}ox^{n-2} +$ &c. which are therefore Exponents of the Ratio of the Increments. Hitherto I have ſuppoſed that x flows, that x hath a real Increment that o is ſomething. And I have proceeded all along on that Suppoſition, without which I ſhould not have been able to have made ſo much as one ſingle Step. From that Suppoſition it is that I get at the Increment of x^n, that I am able to compare it with the Increment of x, and that I find the Proportion between the two Increments. I now beg leave to make a new Suppoſition contrary to the Firſt, *i. e.* I will ſuppoſe that there is no Increment of x, or that o is nothing; which Second Suppoſition deſtroys my firſt, and is inconſiſtent with it, and therefore with every thing that ſuppoſeth it. I do nevertheleſs beg leave to retain $n \times n - 1$, which is an Expreſſion obtained in Virtue of my Firſt Suppoſition, which neceſſarily pre-ſuppoſed ſuch Suppoſition, and which could not be obtained without it. All which ſeems a moſt inconſiſtent Way of arguing, and ſuch as would not be allowed of in Divinity.

XV. Nothing is plainer than that no juſt Concluſion can be directly drawn from two inconſiſtent Suppoſitions. You may indeed ſuppoſe any Thing poſſible; But afterwards you may not ſuppoſe any Thing that deſtroys what you firſt ſuppoſed. Or if you do, you muſt begin *de novo*. If therefore you ſuppoſe that the Augments vaniſh, *i. e.* that there are no Augments, you are to begin again, and ſee what follows from ſuch Suppoſition. But nothing will follow to your Purpoſe. You cannot by that means ever arrive at your Concluſion, or ſucceed in, what is called by the celebrated Author, the Inveſtigation of the firſt or laſt Proportions of naſcent and evaneſcent Quantities, by inſtituting the Analyſis in finite Ones. I repeat it again: You are at Liberty to make any poſſible Suppoſition: And you may deſtroy one Suppoſition by another: But then you may not retain the Conſequences, or any Part of the Conſequences of your firſt Suppoſition ſo deſtroyed. I admit that

Signs may be made to denote either any Thing or Nothing : And conſequently that in the original Notation $x+o$, o might have ſignified either an Increment or nothing. But then which of theſe ſoever you make it ſignify, you muſt argue conſiſtently with ſuch its Signification, and not proceed upon a double Meaning : Which to do were a manifeſt Sophiſm. Whether you argue in Symbols or in Words, the Rules of right Reaſon are ſtill the ſame. Nor can it be ſuppoſed, you will plead a Privilege in Mathematicks to be exempt from them.

XVI. If you aſſume at firſt a Quantity increaſed by nothing, and in the Expreſſion $x+o$, o ſtands for nothing, upon this Suppoſition as there is no Increment of the Root, ſo there will be no Increment of the Power ; and conſequently there will be none except the Firſt, of all thoſe Members of the Series conſtituting the Power of the Binomial ; and will therefore never come to your Expreſſion of a Fluxion legitimately by ſuch Method. Hence you are driven into the fallacious Way of proceeding to a certain Point on the Suppoſition of an Increment, and then at once ſhifting your Suppoſition to that of no Increment. There may ſeem great Skill in doing this at a certain Point or Period. Since if this Second Suppoſition had been made before the common Diviſion by o, all had vaniſhed at once, and you muſt have got nothing by your Suppoſition. Whereas by this Artifice of firſt dividing, and then changing your Suppoſition, you retain 1 and nx^{n-1}. But, notwithſtanding all this Addreſs to cover it, the Fallacy is ſtill the ſame. For whether it be done ſooner or later, when once the Second Suppoſition or Aſſumption is made, in the ſame Inſtant the former Aſſumption and all that you got by it is deſtroyed, and goes out together. And this is univerſally True, be the Subject what it will, throughout all the Branches of humane Knowledge; in any other of which, I believe Men would hardly admit ſuch a reaſoning as this, which in Mathematicks is accepted for Demonſtration.

XVII. It may not be amiſs to obſerve, that the Method for finding the Fluxion of a Rectangle of two flowing Quantities, as it is ſet forth in the Treatiſe of Quadratures, differs from the abovementioned taken from the Second Book of the Principles, and is in Effect the ſame with that uſed in the *calculus differentialis* *. For the ſuppoſing a Quantity infinitely diminiſhed and therefore rejecting it, is in Effect the rejecting an Infiniteſimal; and indeed it requires a marvellous Sharpneſs of Diſcernment, to be able to diſtinguiſh between evaneſcent Increments and infiniteſimal Differences. It may perhaps be ſaid that the Quantity being infinitely diminiſhed becomes nothing, and ſo nothing is rejected. But according to the received Principles it is evident, that no Geometrical Quantity, can by any Diviſion or Subdiviſion whatſoever be exhauſted, or reduced to nothing. Conſidering the various Arts and Devices uſed by the great Author of the Fluxionary Method: in how many Lights he placeth his Fluxions: and in what different Ways he attempts to demonſtrate the ſame Point: one would be inclined to think, he was himſelf ſuſpicious of the Juſtneſs of his own Demonſtrations; and that he was not enough pleaſed with any one Notion ſteadily to adhere to it. Thus much at leaſt is plain, that he owned himſelf ſatisfied concerning certain Points, which nevertheleſs he would not undertake to demonſtrate to others *. Whether this Satisfaction aroſe from tentative Methods or Inductions; which have often been admitted by Mathematicians, (for Inſtance by Dr. *Wallis* in his Arithmetic of Infinites) is what I ſhall not pretend to determine. But, whatever the Caſe might have been with reſpect to the Author, it appears that his Followers have ſhewn themſelves more eager in applying his Method, than accurate in examining his Principles.

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* Analyſe des infiniment petits, part. 1. prop. 2.

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* *See Letter to* Collins, Nov. *8*, *1676*.

XVIII. It is curious to obſerve, what ſubtilty and skill this great Genius employs to ſtruggle with an inſuperable Difficulty; and through what Labyrinths he endeavours to eſcape the Doctrine of Infiniteſimals; which as it intrudes upon him whether he will or no, ſo it is admitted and embraced hy others without the leaſt Repugnance. *Leibnitz* and his Followers in their *calculus differentialis* making no manner of Scruple, Firſt to ſuppoſe, and Secondly to reject Quantities infinitely ſmall: with what clearneſs in the Apprehenſion and juſtneſs in the Reaſoning, any thinking Man, who is not prejudiced in favour of thoſe Things, may eaſily diſcern. The Notion or Idea of an infiniteſimal Quantity, as it is an Object ſimply apprehended by the Mind, hath been already conſidered *. I ſhall now only obſerve as to the Method of getting rid of ſuch Quantities, that it is done without the leaſt Ceremony. As in Fluxions the Point of firſt Importance, and which paves the Way to the Reſt, is to find the Fluxion of a Product of two indeterminate Quantities, ſo in the *calculus differentialis* (which Method is ſuppoſed to have been borrowed from the Former with ſome ſmall Alterations) the main Point is to obtain the Difference of ſuch Product. Now the Rule for this is got by rejecting the

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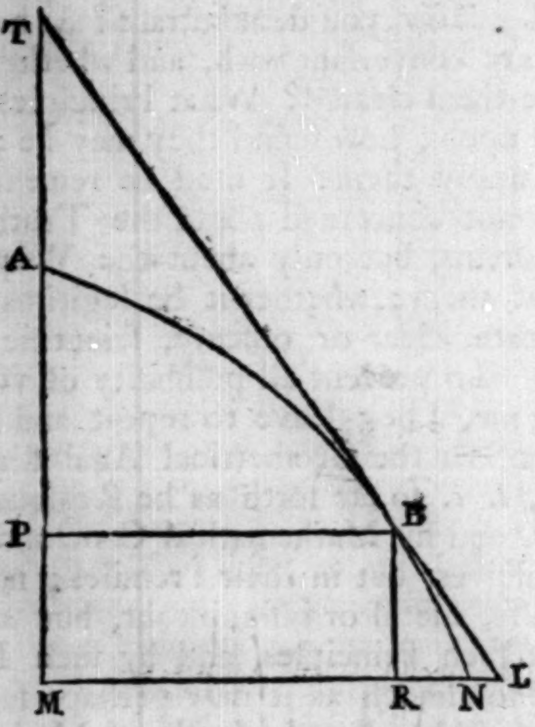
* *Sect.* 5. and 6.

Product or Rectangle of the Differences. And in general it is ſuppoſed, that no Quantity is bigger or leſſer for the Addition or Subduction of its Infiniteſimal: and conſequently no Error can ariſe from ſuch rejection of Infiniteſimals.

XIX. And yet it ſhould ſeem that, whatever Errors are admitted in the Premiſes, proportional Errors ought to be apprehended in the Concluſion, be they finite or infiniteſimal: and therefore the ἀκρίβεια of Geometry requires nothing ſhould be neglected or rejected. In Anſwer to this you will perhaps ſay, that the Concluſions are accurately true, and that therefore the Principles and Methods from whence they are derived muſt be ſo too. But this inverted Way of demonſtrating your Principles by your Concluſions, as it would be peculiar to you Gentlemen, ſo it is contrary to the Rules of Logic. The Truth of the Concluſion will not prove either the Form or the Matter of a Syllogiſm to be true; inaſmuch as the Illation might have been wrong or the Premiſes falſe, and the Concluſion nevertheleſs true, though not in Virtue of ſuch Illation or of ſuch Premiſes. I ſay that in every other Science Men prove their Concluſions by their Principles, and not their Principles by the Concluſions. But if in yours you ſhould allow your ſelves this unnatural Way of proceeding, the Conſequence would be that you muſt take up with Induction, and bid adieu to Demonſtration. And if you ſubmit to this, your Authority will no longer lead the Way in Points of Reaſon and Science.

XX. I have no Controverſy about your Concluſions, but only about your Logic and Method. How you demonſtrate? What Objects you are converſant with, and whether you conceive them clearly? What Principles you proceed upon; how ſound they may be; and how you apply them? It muſt be remembred that I am not concerned about the Truth of your Theorems, but only about the Way of coming at them; whether it be legitimate or illegitimate, clear or obſcure, ſcientific or tentative. To prevent all poſſibility of your miſtaking me, I beg leave to repeat and inſiſt, that I conſider the Geometrical Analyſt as a Logician, *i. e.* ſo far forth as he Reaſons and Argues, and his Mathematical Concluſions, not in themſelves, but in their Premiſes; not as true or falſe, uſeful or inſignificant, but as derived from ſuch Principles, and by ſuch Inferences. And foraſmuch as it may perhaps ſeem an unaccountable Paradox, that Mathematicians ſhould deduce true Propoſitions from falſe Principles, be right in the Concluſion, and yet err in the Premiſes; I ſhall endeavour particularly to explain why this may come to paſs, and ſhew how Error may bring forth Truth, though it cannot bring forth Science.

XXI. In order therefore to clear up this Point, we will ſuppoſe for Inſtance that a Tangent is to be drawn to a Parabola, and examine the Progreſs of this Affair, as it is performed by infiniteſimal Differences.



Let AB be a Curve, the Abſciſſe $AP=x$, the Ordinate $PB=y$, the Difference of the Abſciſſe $PM=dx$, the Difference of the Ordinate $RN=dy$. Now by ſuppoſing the Curve to be a Polygon, and conſequently BN, the Increment or Difference of the Curve, to be a ſtraight Line coincident with the Tangent, and the differential Triangle BRN to be ſimilar to the Triangle TPB the Subtangent PT is found a fourth Proportional to $RN : RB : PB$: that is to $dy : dx : y$. Hence the Subtangent will be $\frac{ydx}{dy}$. But herein there is an Error ariſing from the forementioned falſe Suppoſition, whence the Value of PT comes out greater than the Truth: for in Reality it is not the Triangle RNB but RLB, which is ſimilar to PBT, and therefore (inſtead of RN) RL ſhould have been the firſt Term of the Proportion, *i. e.* $RN + NL$, *i. e.* $dy + z$: whence the true Expreſſion for the Subtangent ſhould have been $\frac{ydx}{dy+z}$. There was therefore an Error of Defect in making dy the Diviſor: which Error was equal to z, *i. e.* NL the Line comprehended between the Curve and the Tangent. Now by the Nature of the Curve $yy = px$, ſuppoſing p to be the Parameter, whence by the Rule of Differences $2ydy = pdx$ and $dy = \frac{pdx}{2y}$. But if you multiply $y + dy$ by it ſelf, and retain the whole Product without rejecting the Square of the Difference, it will then come out, by ſubſtituting the augmented Quantities in the Equation of the Curve, that $dy = \frac{pdx}{2y} - \frac{dydy}{2y}$

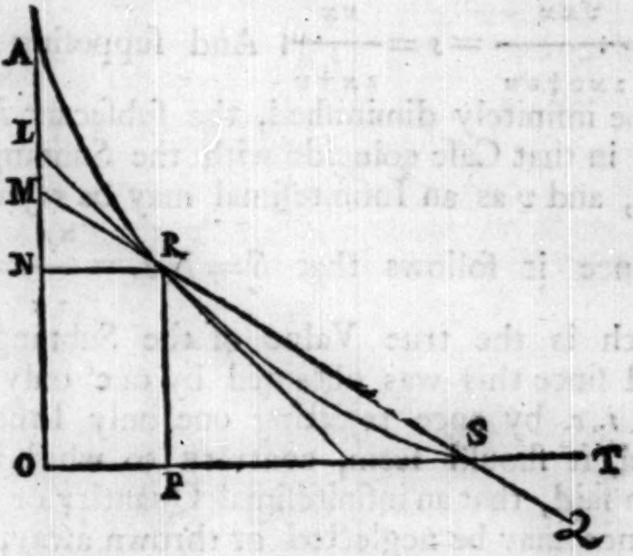
truly. There was therefore an Error of Exceſs in making $dy=\frac{pdx}{2y}$, which followed from the erroneous Rule of Differences. And the Meaſure of this ſecond Error is $\frac{dydy}{2y}=z$. Thereſore the two Errors being equal and contrary deſtroy each other; the firſt Error of Defect being corrected by a ſecond Error of Exceſs.

XXII. If you had committed only one Error, you would not have come at a true Solution of the Problem. But by Virtue of a twofold Miſtake you arrive, though not at Science, yet at Truth. For Science it cannot be called, when you proceed blindfold, and arrive at the Truth not knowing how or by what Means. To demonſtrate that z is equal to $\frac{dydy}{2y}$, let BR or dx be m and RN or dy be n. By the thirty third Propoſition of the firſt Book of the Conics of *Apollonius*, and from ſimilar Triangles, as $2x$ to y ſo is m to $n+z=\frac{my}{2x}$. Likewiſe from the Nature of the Parabola $yy+2yn+nn=xp+mp$, and $2yn+nn=mp$: whereſore $\frac{2yn+nn}{p}=m$; and becauſe

$yy = px$, $\frac{yy}{p}$ will be equal to x. Therefore ſubſtituting theſe Values inſtead of m and x we ſhall have $n + z = \frac{my}{2x} = \frac{2yynp + ynnp}{2yyp}$: i. e. $n + z = \frac{2yn + nn}{2y}$: which being reduced gives $z = \frac{nn}{2y} = \frac{dydy}{2y}$ *Q. E. D.*

XXIII. Now I obſerve in the firſt Place, that the Concluſion comes out right, not becauſe the rejected Square of *dy* was infinitely Small; but becauſe this Error was compenſated by another contrary and equal Error. I obſerve in the ſecond Place, that whatever is rejected, be it ever ſo ſmall, if it be real and conſequently makes a real Error in the Premiſes, it will produce a proportional real Error in the Concluſion. Your Theorems therefore cannot be accurately True, nor your Problems accurately Solved, in Virtue of Premiſes, which themſelves are not accurate, it being a Rule in Logic that *Concluſio ſequitur partem debiliorem.* Therefore I obſerve in the third Place, that when the Concluſion is evident and the Premiſes obſcure, or the Concluſion accurate and the Premiſes inaccurate, we may ſafely pronounce that ſuch Concluſion is neither evident nor accurate, in Virtue of thoſe obſcure inaccurate Premiſes or Principles; but in Virtue of ſome other Principles which perhaps the Demonſtrator himſelf never knew or thought of. I obſerve in the laſt Place, that in Caſe the Differences are ſuppoſed finite Quantities ever ſo great, the Concluſion will nevertheleſs come out the ſame: inaſmuch as the rejected Quantities are legitimately thrown out, not for their ſmallneſs, but for another Reaſon, to wit, becauſe of contrary Errors, which deſtroying each other do upon the whole cauſe that nothing is really, though ſomething is apparently thrown out. And this Reaſon holds equally, with reſpect to Quantities finite as well as infiniteſimal, great as well as ſmall, a Foot or a Yard long as well as the minuteſt Increment.

XXIV, For the fuller Illuſtration of this Point, I ſhall conſider it in another Light, and proceeding in finite Quantities to the Concluſion, I ſhall only then make uſe of one



Infiniteſimal. Suppoſe the ſtraight Line *MQ* cuts the Curve *AT* in the Points *R* and *S*. Suppoſe *LR* a Tangent at the Point *R*, *AN* the Abſciſſe, *NR* and *OS* Ordinates. Let *AN* be produced to *O*, and *RP* be drawn parallel to *NO*. Suppoſe $AN=x$, $NR=y$, $NO=v$, $PS=z$, the ſubſecant $MN=S$. Let the Equation $y=xx$ expreſs the Nature of the Curve: and ſuppoſing y and x increaſed by their finite Increments, we get $y+z=xx+2xv+vv$: whence the former Equation being ſubducted there remains $z=2xv+vv$. And by reaſon of ſimilar Triangles $PS:PR::NR:NM$, i. e. $z:v::y:s=\frac{vy}{z}$, wherein if for y and z we ſubſtitute their Values, we get $\frac{vxx}{2xv+vv}=s=\frac{xx}{2x+v}$. And ſuppoſing NO to be infinitely diminiſhed, the ſubſecant NM will in that Caſe coincide with the Subtangent NL, and v as an Infiniteſimal may be rejected whence it follows that $S=NL=\frac{xx}{2x}=\frac{x}{2}$; which is the true Value of the Subtangent. And ſince this was obtained by one only Error, *i. e.* by once rejecting one only Infiniteſimal, it ſhould ſeem, contrary to what hath been ſaid, that an infiniteſimal Quantity or Difference may be neglected or thrown away, and the Concluſion nevertheleſs be accurately True, although there was no double Miſtake or rectifying of one Error by another, as in the firſt Caſe. But if this Point be throughly conſidered, we ſhall find there is even here a double Miſtake, and that One compenſates or rectifies the Other. For in the firſt Place, it was ſuppoſed, that when NO is infinitely diminiſhed or becomes an Infiniteſimal, then the Subſecant NM becomes equal to the Subtangent NL. But this is a plain Miſtake, for it is evident, that as a Secant cannot be a Tangent, ſo a Subſecant cannot be a Subtangent. Be the Difference ever ſo ſmall, yet ſtill there is a Difference. And if NO be infinitely ſmall, there will even then be an infinitely ſmall Difference between *N M* and *N L*. Therefore *N M* or *S* was too little for your Suppoſition, (when you ſuppoſed it equal to *N L*) and this Error was compenſated by a ſecond Error in throwing out *v*, which laſt Error made *s* bigger than its true Value, and in lieu thereof gave the Value of the Subtangent. This is the true State of the Caſe, however it may be diſguiſed. And to this in reality it amounts, and is at bottom the ſame Thing, if we ſhould pretend to find the Subtangent by having firſt found, from the Equation of the Curve and ſimilar Triangles, a general Expreſſion for all Subſecants, and then reducing the Subtangent under this general Rule, by conſidering it as the Subſecant when *v* vaniſhes or becomes nothing.

XXV. Upon the whole I obſerve, *Firſt*, that *v* can never be nothing ſo long as there is a Secant. *Secondly*, That the ſame Line cannot be both Tangent and Secant. *Thirdly*, that when *v* or *NO* * vaniſheth, *P S* and *S R* do alſo vaniſh, and with them the proportionality of the ſimilar Triangles. Conſequently the whole Expreſſion, which was obtained by means thereof and grounded thereupon, vaniſheth when *v* vaniſheth. *Fourthly*, that the Method for finding Secants or the Expreſſion of Secants, be it ever ſo general, cannot in common

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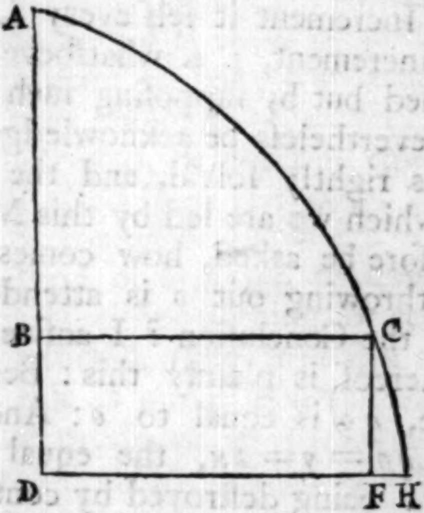
* *See the foregoing Figure.*

Senſe extend any further than to all Secants whatſoever: and, as it neceſſarily ſuppoſed ſimilar Triangles, it cannot be ſuppoſed to take Place where there are not ſimilar Triangles. *Fifthly*, that the Subſecant will always be leſs than the Subtangent, and can never coincide with it; which Coincidence to ſuppoſe would be abſurd; for it would be ſuppoſing, the ſame Line at the ſame Time to cut and not to cut another given Line, which is a manifeſt Contradiction, ſuch as ſubverts the Hypotheſis and gives a Demonſtration of its Falſhood. *Sixthly*, If this be not admitted, I demand a Reaſon why any other apagogical Demonſtration, or Demonſtration *ad abſurdum* ſhould be admitted in Geometry rather than this: Or that ſome real Difference be aſſigned between this and others as ſuch. *Seventhly*, I obſerve that it is ſophiſtical to ſuppoſe NO or RP, PS, and SR to be finite real Lines in order to form the Triangle RPS, in order to obtain Proportions by ſimilar Triangles; and afterwards to ſuppoſe there are no ſuch Lines, nor conſequently ſimilar Triangles, and nevertheleſs to retain the Conſequence of the Firſt Suppoſition, after ſuch Suppoſition hath been deſtroyed by a contrary one. *Eighthly*, That although, in the preſent Caſe, by inconſiſtent Suppoſitions Truth may be obtained, yet that ſuch Truth is not demonſtrated: That ſuch Method is not conformable to the Rules of Logic and right Reaſon: That, however uſeful it may be, it muſt be conſidered only as a Preſumption, as a Knack, an Art rather an Artifice, but not a ſcientific Demonſtration.

XXVI. The Doctrine premiſed may be farther illuſtrated by the following ſimple and eaſy Caſe, wherein, I ſhall proceed by evaneſcent Increments. Suppoſe $AB = x$, $BC = y$, $BD = o$, and that xx is equal to the Area ABC; It is propoſed to find the Ordinate y or BC. When x by flowing becomes $x + o$, then xx becomes $xx + 2xo + oo$: And the Area ABC becomes ADH, and the Increment of xx will be equal to $BDHC$ the Increment of the



Area, *i. e.* to $BCFD+CFH$. And if we ſuppoſe the curvilinear Space CFH to be qoo, then $2xo+oo=yo+qoo$ which divided by o gives $2x+o=y+qo$. And, ſuppoſing o to vaniſh, $2x=y$, in which Caſe ACH will be a ſtraight Line, and the Areas ABC, CFH, Triangles. Now with regard to this Reaſoning, it hath been already remarked *, that it is not legitimate or logical to ſuppoſe o to vaniſh, *i. e.* to be nothing, *i. e.* that there is no Increment, unleſs we reject at the ſame Time with the Increment it ſelf every Conſequence of ſuch Increment, *i. e.* whatſoever could not be obtained but by ſuppoſing ſuch Increment. It muſt nevertheleſs be acknowledged, that the Problem is rightly ſolv'd, and the Concluſion true, to which we are led by this Method. It will therefore be asked, how comes it to paſs that the throwing out o is attended with no Error in the Concluſion? I anſwer, the true Reaſon hereof is plainly this: Becauſe q being Unite, qo is equal to o: And therefore $2x+o-qo=y=2x$, the equal Quantities qo, and o being deſtroyed by contrary Signs.

XXVII. As on the one Hand it were abſurd to get rid of o by ſaying, let me contradict my ſelf: Let me ſubvert my own Hypotheſis: Let me take it for granted that there is no Increment, at the ſame Time that I retain a Quantity, which I could never have got at but by aſſuming an Increment: So on the other Hand it would be equally wrong to imagine, that in a Geometrical Demonſtration we may be allowed to admit any Error, though ever ſo ſmall, or that it is poſſible, in the Nature of Things, an accurate Concluſion ſhould be derived from inaccurate Principles. Therefore o cannot be thrown out as an Infiniteſimal, or upon the Principle that Infiniteſimals may be ſafely neglected. But only becauſe it is deſtroyed by an equal Quantity with a negative Sign, whence $o-po$ is equal to nothing. And as it is illegitimate to reduce an Equation, by ſubducting from one Side a Quantity when it is not to be deſtroyed, or when an equal Quantity is not ſubducted from the other Side of the Equation: So it muſt be allowed a very logical and juſt Method of arguing, to conclude that if from Equals either nothing or equal Quantities are ſubducted, they ſhall ſtill remain equal. And this is a true Reaſon why no Error is at laſt produced by the rejecting of o. Which therefore muſt not be aſcribed to the Doctrine of Differences, or Infiniteſimals, or evaneſcent Quantities, or Momemtums, or Fluxions.

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* *Sect.* 12. *and* 13. ſupra.

XXVIII. Suppoſe the Caſe to be general, and that x^n is equal to the Area ABC, whence by the Method of Fluxions the Ordinate is found nx^{n-1}, which we admit for true, and ſhall inquire how it is arrived at. Now if we are content to come at the Concluſion in a ſummary Way, by ſuppoſing that the Ratio of the Fluxions of x and x^n are found * to be 1 and nx^{n-1}, and that the Ordinate of the Area is conſidered as its Fluxion; we ſhall not ſo clearly ſee our Way, or perceive how the Truth comes out, that Method as we have ſhewed before being obſcure and illogical. But if we fairly delineate the Area and its Increment, and divide the Latter into two Parts $BCFD$ and CFH †, and proceed regularly by Equations between the Algebraical and Geometrical Quantities, the Reaſon of the Thing will plainly appear. For as x^n is equal to the Area ABC, ſo is the Increment of x^n equal to the Increment of the Area, *i. e.* to $BDHC$; that is, to ſay, $nox^{n-1} + \frac{nn-n}{2} oox^{n-2} +$ &c. $= BDFC + CFH$. And only the Firſt Members, on each Side of the Equation being retained, $nox^{n-1} = BDFC$: And dividing both Sides by o or BD, we ſhall get $nx^{n-1} = BC$. Admitting, therefore, that the curvilinear Space CFH is equal to the rejectaneous Quantity $\frac{nn-n}{2} oox^{n-2} +$ &c. and that when this is rejected on one Side, that is rejected on the Other, the Reaſoning becomes juſt and the Concluſion true. And it is all one whatever Magnitude you allow to BD, whether that of an infiniteſimal Difference or a finite Increment ever ſo Great. It is therefore Plain, that the ſuppoſing the rejectaneous Algebraical Quantity to be an infinitely ſmall or evaneſcent Quantity, and therefore to be neglected, muſt have produced an Error, had it not been for the curvilinear Spaces being equal thereto, and at the ſame Time ſubducted from the other Part or Side of the Equation agreeably to the Axiom. *If from Equals you ſubduct Equals, the Remainders will be equal.* For thoſe Quantities which by the Analyſts are ſaid to be neglected, or made to vaniſh, are in reality ſubducted. If therefore the Concluſion be true, it is abſolutely neceſſary that the finite Space CFH be equal to the Remainder of the Increment expreſſed by $\frac{nn-n}{2}oox^{n-2}$ &c. equal I ſay to the finite Remainder of a finite Increment.

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* *Sect.* 13.

† *See the Figure in Sect.* 26.

XXIX. Therefore, be the Power what you pleaſe, there will ariſe on one Side an Algebraical Expreſſion, on the Other a Geometrical Quantity, each of which naturally divides it ſelf into three Members: The Algebraical or fluxionary Expreſſion, into one which includes neither the Expreſſion of the

Increment of the Abſciſs nor of any Power thereof, another which includes the Expreſſion of the Increment it ſelf, and a third including the Expreſſion of the Powers of the Increment. The Geometrical Quantity alſo or whole increaſed Area conſiſts of three Parts or Members, the Firſt of which is the given Area, the Second a Rectangle under the Ordinate and the Increment of the Abſciſs, and the the Third a curvilinear Space. And, comparing the homologous or correſpondent Members on both Sides, we find that as the Firſt Member of the Expreſſion is the Expreſſion of the given Area, ſo the Second Member of the Expreſſion will expreſs the Rectangle or ſecond Member of the Geometrical Quantity; and the Third, containing the Powers of the Increment, will expreſs the curvilinear Space, or third Member of the Geometrical Quantity. This Hint may, perhaps, be further extended and applied to good Purpoſe, by thoſe who have Leiſure and Curioſity for ſuch Matters. The Uſe I make of it is to ſhew, that the Analyſis cannot obtain in Augments or Differences, but it muſt alſo obtain in finite Quantities, be they ever ſo great, as was before obſerved.

XXX. It ſeems therefore upon the whole that we may ſafely pronounce, the Concluſion cannot be right, if in order thereto any Quantity be made to vaniſh, or be neglected, except that either one Error is redreſſed by another; or that Secondly, on the ſame Side of an Equation equal Quantities are deſtroyed by contrary Signs, ſo that the Quantity we mean to reject is firſt annihilated; or laſtly, that from the oppoſite Sides equal Quantities are ſubducted. And therefore to get rid of Quantities by the received Principles of Fluxions or of Differences is neither good Geometry nor good Logic. When the Augments vaniſh, the Velocities alſo vaniſh. The Velocities or Fluxions are ſaid to be *primò* and *ultimò*, as the Augments naſcent and evaneſcent. Take therefore the *Ratio* of the evaneſcent Quantities, it is the ſame with that of the Fluxions. It will therefore anſwer all Intents as well. Why then are Fluxions introduced? Is it not to ſhun or rather to palliate the Uſe of Quantities infinitely ſmall? But we have no Notion whereby to conceive and meaſure various Degrees of Velocity, beſides Space and Time, or when the Times are given, beſides Space alone. We have even no Notion of Velocity preſcinded from Time and Space. When therefore a Point is ſuppoſed to move in given Times, we have no Notion of greater or leſſer Velocities or of Proportions between Velocities, but only of longer or ſhorter Lines, and of Proportions between ſuch Lines generated in equal Parts of Time.

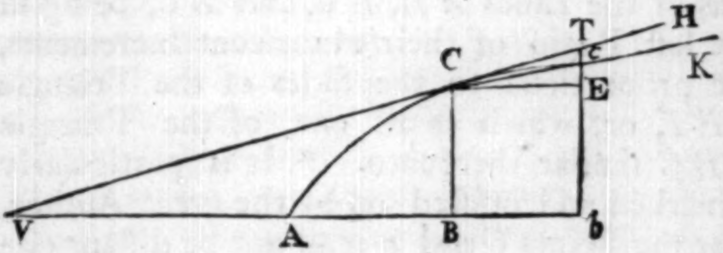
XXXI. A Point may be the Limit of a Line: A Line may be the Limit of a Surface: A Moment may terminate Time. But how can we conceive a Velocity by the help of ſuch Limits? It neceſſarily implies both Time and Space, and cannot be conceived without them. And if the Velocities of naſcent and evaneſcent Quantities, *i. e.* abſtracted from Time and Space, may not be comprehended, how can we comprehend and demonſtrate their Proportions? Or conſider their *rationes primæ* and *ultimæ*. For to conſider the Proportion or *Ratio* of Things implies that ſuch Things have Magnitude: That ſuch their Magnitudes may be meaſured, and their Relations to each other known. But, as there is no meaſure of Velocity except Time and Space, the Proportion of Velocities being only compounded of the direct Proportion of the Spaces, and the reciprocal Proportion of the Times; doth it not follow that to talk of inveſtigating, obtaining, and conſidering the Proportions of Velocities, excluſively of Time and Space, is to talk unintelligibly?

XXXII. But you will ſay that, in the Uſe and Application of Fluxions, Men do not overſtrain their Faculties to a preciſe Conception of the abovementioned Velocities, Increments, Infiniteſimals, or any other ſuch like

Ideas of a Nature ſo nice, ſubtil, and evaneſcent. And therefore you will perhaps maintain, that Problems may be ſolved without thoſe inconceivable Suppoſitions: and that, conſequently, the Doctrine of Fluxions, as to the practical Part, ſtands clear of all ſuch Difficulties. I anſwer, that if in the Uſe or Application of this Method, thoſe difficult and obſcure Points are not attended to, they are nevertheleſs ſuppoſed. They are the Foundations on which the Moderns build, the Principles on which they proceed, in ſolving Problems and diſcovering Theorems. It is with the Method of Fluxions as with all other Methods, which preſuppoſe their reſpective Principles and are grounded thereon. Although the Rules may be practiſed by Men who neither attend to, nor perhaps know the Principles. In like manner, therefore, as a Sailor may practically apply certain Rules derived from Aſtronomy and Geometry, the Principles whereof he doth not underſtand: And as any ordinary Man may ſolve divers numerical Queſtions, by the vulgar Rules and Operations of Arithmetic, which he performs and applies without knowing the Reaſons of them: Even ſo it cannot be denied that you may apply the the Rules of the Fluxionary Method: You may compare and reduce particular Caſes to general Forms: You may operate and compute and ſolve Problems thereby, not only without an actual Attention to, or an actual Knowledge of, the Grounds of that Method, and the Principles whereon it depends, and whence it is deduced, but even without having ever conſidered or comprehended them.

XXXIII. But then it muſt be remembred, that in ſuch Caſe although you may paſs for an Artiſt, Computiſt, or Analyſt, yet you may not be juſtly eſteemed a Man of Science and Demonſtration. Nor ſhould any Man, in virtue of being converſant in ſuch obſcure Analytics, imagine his rational Faculties to be more improved than thoſe of other Men, which have been exerciſed in a different manner, and on different Subjects; much leſs erect himſelf into a Judge and an Oracle, concerning Matters that have no ſort of Connexion with, or Dependence on thoſe Species, Symbols or Signs, in the Management whereof he is ſo converſant and expert. As you, who are a skilful Computiſt or Analyſt, may not therefore be deemed skilful in Anatomy: or *vice verſa*, as a Man who can diſſect with Art, may, nevertheleſs, be ignorant in your Art of Computing: Even ſo you may both, notwithſtanding your peculiar Skill in your reſpective Arts, be alike unqualified to decide upon Logic, or Metaphyſics, or Ethics, or Religion. And this would be true, even admitting that you underſtood your own Principles and could demonſtrate them.

XXXIV. If it is ſaid, that Fluxions may be expounded or expreſſed by finite Lines proportional to them: Which finite Lines, as they may be diſtinctly conceived and known and reaſoned upon, ſo they may be ſubſtituted for the Fluxions, and their mutual Relations or Proportions be conſidered as the Proportions of Fluxions: By which means the Doctrine becomes clear and uſeful. I anſwer that if, in order to arrive at theſe finite Lines proportional to the Fluxions, there be certain Steps made uſe of which are obſcure and inconceivable, be thoſe finite Lines themſelves ever ſo clearly conceived, it muſt neverthele ſs be acknowledged, that your proceeding is not clear nor your Method ſcientific. For Inſtance, it is ſuppoſed that *A B* being the Abſciſs, *B C* the Ordinate, and *V C H* a Tangent of the Curve *A C*, *B b* or *C E* the Increment of the Abſciſs, *E c* the Increment of the Ordinate which produced meets *V H* in the Point *T*, and *C c* the Increment of the Curve. The right Line



Line *C c* being produced to *K*, there are formed three ſmall Triangles, the Rectilinear *C E c* the Mixtilinear *C E c*, and the Rectilinear Triangle *C E T*. It is evident theſe three Triangles are different from each other, the Rectilinear *C E c* being leſs than the Mixtilinear *C E c*, whoſe Sides are the three Increments abovementioned, and this ſtill leſs than the Triangle *C E T*. It is ſuppoſed that the Ordinate *b c* moves into the place *B C*, ſo that the Point *c* is coincident with the Point *C*; and the right Line *C K*, and conſequently the Curve *C c*, is coincident with the Tangent *C H*. In which caſe the mixtilinear evaneſcent Triangle *C E c* will, in its laſt Form, be ſimilar to the Triangle *C E T*: And its evaneſcent Sides *C E*, *E c*, and *C c* will be proportional to *C E*, *E T*, and *C T* the Sides of the Triangle *C E T*. And therefore it is concluded, that the Fluxions of the Lines *A B*, *B C*, and *A C*, being in the laſt Ratio of their evaneſcent Increments, are proportional to the Sides of the Triangle *C E T*, or, which is all one, of the Triangle *V B C* ſimilar thereunto. * It is particularly remarked and inſiſted on by the great Author, that the Points *C* and *c* muſt not be diſtant one from another, by any the leaſt Interval whatſoever: But that, in order to find the ultimate Proportions of the Lines *C E*, *E c*, and *C c* (*i. e.* the Proportions of the Fluxions or Velocities)

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* Introd. ad Quadraturam Curvarum.

locities) expreſſed by the finite Sides of the Triangle *V B C*, the Points *C* and *c* muſt be accurately coincident, *i. e.* one and the ſame. A Point therefore is conſidered as a Triangle, or a Triangle is ſuppoſed to be formed in a Point. Which to conceive ſeems quite impoſſible. Yet ſome there are, who, though they ſhrink at all other Myſteries, make no Difficulty of their own, who ſtrain at a Gnat and ſwallow a Camel.

XXXV. I know not whether it be worth while to obſerve, that poſſibly ſome Men may hope to operate by Symbols and Suppoſitions, in ſuch ſort as to avoid the Uſe of Fluxions, Momentums, and Infiniteſimals after the following manner. Suppoſe x to be one Abſciſs of a Curve, and z another Abſciſs of the ſame Curve. Suppoſe alſo that the reſpective Areas are xxx and zzz: and that $z-x$ is the Increment of the Abſciſs, and $zzz-xxx$ the Increment of the Area, without conſidering how great, or how ſmall thoſe Increments may be. Divide now $zzz-xxx$ by $z-x$ and the Quotient will be $zz+zx+xx$: and, ſuppoſing that z and x are equal, this ſame Quotient will be $3xx$ which in that caſe is the Ordinate, which therefore may be thus obtained independently of Fluxions and Infiniteſimals. But herein is a direct Fallacy: for in the firſt place, it is ſuppoſed that the Abſciſſes z and x are unequal, without which Suppoſition no one ſtep could have been made; and in the ſecond place, it is ſuppoſed they are equal; which is a manifeſt Inconſiſtency, and amounts to the ſame thing that hath been before conſidered *. And there is indeed Reaſon to apprehend, that all Attempts for ſetting the abſtruſe and fine Geometry on a right Foundation, and avoiding the Doctrine of Velocities, Momentums, *&c.* will be found impracticable, till ſuch time as the Object and End of Geometry are better underſtood, than hitherto they ſeem to have been. The great Author of the Method of Fluxions felt this Difficulty, and therefore he gave into thoſe nice Abſtractions and Geometrical Metaphyſics, without which he ſaw nothing could be done on the received Principles; and what in the way of Demonſtration he hath done with them the Reader will judge. It muſt, indeed, be acknowledged, that he uſed Fluxions, like the Scaffold of a Building, as things to be laid aſide or got rid of, as ſoon as finite Lines were found proportional to them. But then theſe finite Exponents are found by the help of Fluxions. Whatever therefore is got by ſuch Exponents and Proportions is to be aſcribed to Fluxions: which muſt therefore be previouſly underſtood. And what are theſe Fluxions? The Velocities of evaneſcent Increments? And what are theſe ſame evaneſcent Increments? They are neither finite Quantities, nor Quantities infinitely ſmall,

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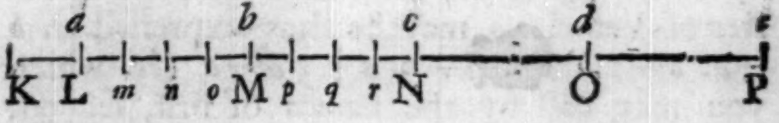
* Sect. 15.

nor

not yet nothing. May we not call them the Ghoſts of departed Quantities?

XXXVI. Men too often impoſe on themſelves and others, as if they conceived and underſtood things expreſſed by Signs, when in truth they have no Idea, ſave only of the very Signs themſelves. And there are ſome grounds to apprehend that this may be the preſent Caſe. The Velocities of evaneſcent or naſcent Quantities are ſuppoſed to be expreſſed, both by finite Lines of a determinate Magnitude, and by Algebraical Notes or Signs: but I ſuſpect that many who, perhaps never having examined the matter, take it for granted, would upon a narrow Scrutiny find it impoſſible, to frame any Idea or Notion whatſoever of thoſe Velocities, excluſive of ſuch finite Quantities and Signs.



Suppoſe the Line *K P* deſcribed by the Motion of a Point continually accelerated, and that in equal Particles of Time the unequal Parts *K L*, *L M*, *M N*, *N O* &c. are generated. Suppoſe alſo that *a*, *b*, *c*, *d*, *e*, &c. denote the Velocities of the generating Point, at the ſeveral Periods of the Parts or Increments ſo generated. It is eaſy to obſerve that theſe

Increments are each proportional to the ſum of the Velocities with which it is deſcribed: That, conſequently, the ſeveral Sums of the Velocities, generated in equal Parts of Time, may be ſet forth by the reſpective Lines *K L*, *L M*, *M N*, &c. generated in the ſame Times: It is likewiſe an eaſy matter to ſay, that the laſt Velocity generated in the firſt Particle of Time, may be expreſſed by the Symbol *a*, the laſt in the ſecond by *b*, the laſt generated in the third by *c*, and ſo on: that *a* is the Velocity of *L M* in *ſtatu naſcenti*, and *b*, *c*, *d*, *e*, &c. are the Velocities of the Increments *M N*, *N O*, *O P*, &c. in their reſpective naſcent Eſtates. You may proceed, and conſider theſe Velocities themſelves as flowing or increaſing Quantities, taking the Velocities of the Velocities, and the Velocities of the Velocities of the Velocities, *i. e.* the firſt, ſecond, third, &c. Velocities *ad infinitum*: which ſucceeding Series of Velocities may be thus expreſſed. a. $b-a$. $c-2b+a$. $d-3c+3b-a$ &c. which you may call by the names of firſt, ſecond, third, fourth Fluxions. And for an apter Expreſſion you may denote the variable flowing Line *K L*, *K M*, *K N*, &c. by the Letter x; and the firſt Fluxions by $\dot{x}$, the ſecond by $\ddot{x}$, the third by $\dddot{x}$, and ſo on *ad infinitum*.

XXXVII. Nothing is eaſier than to aſſign Names, Signs, or Expreſſions to theſe Fluxions, and it is not difficult to compute and operate by means of ſuch Signs. But it will be found much more difficult, to omit the Signs and yet retain in our Minds the things, which we ſuppoſe to be ſignified by them. To conſider the Exponents, whether Geometrical, or Algebraical, or Fluxionary, is no difficult Matter. But to form a preciſe Idea of a third Velocity for Inſtance, in it ſelf and by it ſelf, *Hoc opus, hic labor.* Nor indeed is it an eaſy point, to form a clear and diſtinct Idea of any Velocity at all, excluſive of and preſcinding from all Length of Time and Space; as alſo from all Notes, Signs or Symbols whatſoever. This, if I may be allowed to judge of others by my ſelf, is impoſſible. To me it ſeems evident, that Meaſures and Signs are abſolutely neceſſary, in order to conceive or reaſon about Velocities; and that, conſequently, when we think to conceive the Velocities, ſimply and in themſelves, we are deluded by vain Abſtractions.

XXXVIII. It may perhaps be thought by ſome an eaſier Method of conceiving Fluxions, to ſuppoſe them the Velocities wherewith the infiniteſimal Differences are generated. So that the firſt Fluxions ſhall be the Velocities of the firſt Differences, the ſecond the Velocities of the ſecond Differences, the third Fluxions the Velocities of the third Differences, and ſo on *ad infinitum*. But not to mention the inſurmountable Difficulty of admitting or conceiving Infiniteſimals, and Infiniteſimals of Infiniteſimals, &c. it is evident that this Notion of Fluxions would not conſiſt with the great Author's View; who held that the minuteſt Quantity ought not to be neglected, that therefore the Doctrine of Infiniteſimal Differences was not to be admitted in Geometry, and who plainly appears to have introduced the Uſe of Velocities or Fluxions, on purpoſe to exclude or do without them.

XXXIX. To others it may poſſibly ſeem, that we ſhould form a juſter Idea of Fluxions, by aſſuming the finite unequal Iſochronal Increments *K L*, *L M*, *M N*, &c. and conſidering them in *ſtatu naſcenti*, alſo their Increments in *ſtatu naſcenti*, and the naſcent Increments of thoſe Increments, and ſo on, ſuppoſing the firſt naſcent Increments to be proportional to the firſt Fluxions or Velocities, the naſcent Increments of thoſe Increments to be proportional to the ſecond Fluxions, the third naſcent Increments to be proportional to the third Fluxions, and ſo onwards. And, as the firſt Fluxions are the Velocities of the firſt naſcent Increments, ſo the the ſecond Fluxions may be conceived to be the Velocities of the ſecond naſcent Increments, rather than the Velocities of Velocities. By which means the Analogy of Fluxions may ſeem better preſerved, and the Notion rendered more intelligible.

XL. And indeed it ſhould ſeem, that in the way of obtaining the ſecond or third Fluxion of an Equation, the given Fluxions were conſidered rather as Increments than Velocities. But the conſidering them ſometimes in one Senſe, ſometimes in another, one while in themſelves, another in their Exponents, ſeems to have occaſioned no ſmall ſhare of that Confuſion and Obſcurity, which is found in the Doctrine of Fluxions. It may ſeem therefore, that the Notion might be ſtill mended, and that inſtead of Fluxions of Fluxions, or Fluxions of Fluxions of Fluxions, and inſtead of ſecond, third, or fourth, &c. Fluxions of a given Quantity, it might be more conſiſtent and leſs liable to Exception to ſay, the Fluxion of the firſt naſcent Increment, *i. e.* the ſecond Fluxion; the Fluxion of the ſecond naſcent Increment, *i. e.* the third Fluxion; the Fluxion of the third naſcent Increment, *i. e.* the fourth Fluxion, which Fluxions are conceived reſpectively proportional, each to the naſcent Principle of the Increment ſucceeding that whereof it is the Fluxion.

XLI. For the more diſtinct Conception of all which it may be conſidered, that if the finite Increment *L M*[*] be divided into the Iſochronal Parts *L m*, *m n*, *n o*, *o M*; and the Increment *M N* into the Parts *M p*, *p q*, *q r*, *r N* Iſochronal to the former; as the whole Increments *L M*, *M N* are proportional to the Sums of their deſcribing Velocities, even ſo the homologous Particles *L m*, *M p* are alſo proportional to the reſpective accelerated Velocities with which they are deſcribed. And as the Velocity with which *M p* is generated, exceeds that with which *L m* was generated, even ſo the Particle *M p* exceeds the Particle *L m*. And in general, as the Iſochronal Velocities deſcribing the Particles of *M N* exceed the Iſochronal Velocities deſcribing the Particles of *L M*, even ſo the Particles of the former exceed the correſpondent Particles of the latter. And this will hold, be the ſaid Particles ever ſo ſmall. *M N* therefore will exceed *L M* if they are both taken in their naſcent States: and that Exceſs will be proportional to the Exceſs of the Velocity *b* above the Velocity *a*. Hence we may ſee that this laſt account of Fluxions comes, in the upſhot, to the ſame thing with the firſt[†].

XLII. But notwithſtanding what hath been ſaid it muſt ſtill be acknowledged, that the finite Particles *L m* or *M p*, though taken ever ſo ſmall, are not proportional to the Velocities *a* and *b*; but each to a Series of Velocities changing every Moment, or which is the ſame thing, to an accelerated Velocity, by which it is generated, during a certain minute Particle of Time: That the naſcent Beginnings or evaneſcent Endings of finite Quantities, which are produced in Moments or infinitely ſmall Parts of Time, are alone proportional to given Velocities: That, therefore, in order to conceive the firſt Fluxions, we muſt conceive Time divided into Moments, Increments generated in thoſe Moments, and Velocities proportional to thoſe Increments: That in order to conceive ſecond and third Fluxions, we muſt ſuppoſe that the naſcent Principles or momentaneous Increments have themſelves alſo other momentaneous Increments, which are proportional to their reſpective generating Velocities: That the Velocities of theſe ſecond momentaneous Increments are ſecond Fluxions: thoſe of their naſcent momentaneous Increments third Fluxions. And ſo on *ad infinitum.*

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[*] *See the foregoing Scheme in Sect. 36.*

[†] *Sect. 36.*

XLIII. By ſubducting the Increment generated in the firſt Moment from that generated in the ſecond, we get the Increment of an Increment. And by ſubducting the Velocity generating in the firſt Moment from that generating in the ſecond, we get the Fluxion of a Fluxion. In like manner, by ſubducting the Difference of the Velocities generating in the two firſt Moments, from the Exceſs of the Velocity in the third above that in the ſecond Moment, we obtain the third Fluxion. And after the ſame Analogy we may proceed to fourth, fifth, ſixth Fluxions, *&c.* And if we call the Velocities of the firſt, ſecond, third, fourth Moments *a*, *b*, *c*, *d*, the Series of Fluxions will be as above, $a.\ b-a.\ c-2b+a.\ d-3c+3b-a.$ *ad infinitum*, *i. e.* $\dot{x}.\ \ddot{x}.\ \dddot{x}.\ \ddddot{x}.$ *ad infinitum.*

XLIV. Thus Fluxions may be conſidered in ſundry Lights and Shapes, which ſeem all equally difficult to conceive. And indeed, as it is impoſſible to conceive Velocity without Time or Space, without either finite Length or finite Duration[†], it muſt ſeem above the Powers of Men to comprehend even the firſt Fluxions. And if the firſt are incomprehenſible, what ſhall we ſay of the ſecond and third Fluxions, *&c?* He who can conceive the Beginning of a Beginning, or the End of an End, ſomewhat before the firſt or after the laſt, may be perhaps ſharpſighted enough to conceive theſe things. But moſt Men will, I believe, find it impoſſible to underſtand them in any Senſe whatever.

XLV. One would think that Men could not ſpeak too exactly on ſo nice a Subject. And yet, as was before hinted, we may often obſerve that the Exponents of Fluxions or Notes repreſenting Fluxions are confounded with the Fluxions themſelves. Is not this the Caſe, when juſt after the Fluxions of flowing Quantities were ſaid to be the Celerities of their increaſing, and the ſecond Fluxions to be the Mutations of the firſt Fluxions or Celerities, we are told that $\overset{\prime\prime}{z}.\ \overset{\prime}{z}.\ z.\ \dot{z}.\ \ddot{z}.\ \dddot{z}.$ * repreſents a Series of Quantities, whereof each ſubſequent Quantity is the Fluxion of the preceding; and each foregoing is a fluent Quantity having the following one for its Fluxion?

[†] *Sect.* 31.

XLVI. Divers Series of Quantities and Expreſſions, Geometrical and Algebraical, may be eaſily conceived, in Lines, in Surfaces, in Species, to be continued without End or Limit. But it will not be found ſo eaſy to conceive a Series, either of mere Velocities or of mere naſcent Increments, diſtinct therefrom and correſponding thereunto. Some perhaps may be led to think the Author intended a Series of Ordinates, wherein each Ordinate was the Fluxion of the preceding and Fluent of the following, *i. e.* that the Fluxion of one Ordinate was it ſelf the Ordinate of another Curve; and the Fluxion of this laſt Ordinate was the Ordinate of yet another Curve; and ſo on *ad infinitum.* But who can conceive how the Fluxion (whether Velocity or naſcent Increment) of an Ordinate? Or more than that each preceding Quantity or Fluent is related to its Subſequent or Fluxion, as the Area of curvilinear Figure to its Ordinate; agreeably to what the Author remarks, that each preceding Quantity in ſuch Series is as the Area of a curvilinear Figure, whereof the Abſciſs is *z*, and the Ordinate is the following Quantity.

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* De *Quadratura Curvarum.*

XLVII. Upon the whole it appears that the Celerities are diſmiſſed, and inſtead thereof Areas and Ordinates are introduced. But however expedient ſuch Analogies or ſuch Expreſſions may be found for facilitating the modern Quadratures, yet we ſhall not find any Light given us thereby into the original real Natute of Fluxions; or that we are enabled to frame from thence juſt Ideas of Fluxions conſidered in themſelves. In all this the general ultimate Driſt of the Author is very clear, but his Principles are obſcure. But perhaps thoſe Theories of the great Author are not minutely conſidered or canvaſſed by his Diſciples: who ſeem eager, as was before hinted, rather to operate than to know, rather to apply his Rules and his Forms, than to underſtand his Principles and enter into his Notions. It is nevertheleſs certain, that in order to follow him in his Quadratures, they muſt find Fluents from Fluxions; and in order to this, they muſt know to find Fluxions from Fluents; and in order to find Fluxions, they muſt firſt know what Fluxions are. Otherwiſe they proceed without Clearneſs and without Science. Thus the direct Method precedes the Inverſe, and the Knowledge of the Principles is ſuppoſed in both. But as for operating according to Rules, and by the help of general Forms, whereof the original Principles and Reaſons are not underſtood, this is to be eſteemed merely technical. Be the Principles therefore ever ſo abſtruſe and metaphyſical, they muſt be ſtudied by whoever would comprehend the Doctrine of Fluxions. Nor can any Geometrician have a Right to apply the Rules of the great Author, without firſt conſidering his metaphyſical Notions whence they were derived. Theſe how neceſſary ſoever in order to Science, which can never be attained without a preciſe, clear, and accurate Conception of the Principles, are nevertheleſs by ſeveral careleſly paſſed over; while the Expreſſions alone are dwelt on and conſidered and treated with great Skill and Management, thence to obtain other Expreſſions by Methods, ſuſpicious and indirect (to ſay the leaſt) if conſidered in themſelves, however recommended by Induction and Authority; two Motives which are acknowledged ſufficient to beget a rational Faith and moral Perſwaſion, but nothing higher.

XLVIII. You may poſſibly hope to evade the Force of all that hath been ſaid, and to ſcreen falſe Principles and inconſiſtent Reaſonings, by a general Pretence that theſe Objections and Remarks are Metaphyſical. But this is a vain Pretence. For the plain Senſe and Truth of what is advanced in the foregoing Remarks, I appeal to the Underſtanding of every unprejudiced intelligent Reader. To the ſame I appeal, whether the Points remarked upon are not moſt incomprehenſible Metaphyſics. And Metaphyſics not of mine, but your own. I would not be underſtood to infer, that your Notions are falſe or vain becauſe they are Metaphyſical. Nothing is either true or falſe for that Reaſon. Whether a Point be called Metaphyſical or no avails little. The Queſtion is whether it be clear or obſcure, right or wrong, well or ill-deduced?

XLIX. Although momentaneous Increments, naſcent and evaneſcent Quantities, Fluxions and Infiniteſimals of all Degrees, are in truth ſuch ſhadowy Entities, ſo difficult to imagine or conceive diſtinctly, that (to ſay the leaſt) they cannot be admitted as Principles or Objects of clear and accurate Science: and although this obſcurity and incomprehenſibility of your Metaphyſics had been alone ſufficient, to allay your Pretenſions to Evidence; yet it hath, if I miſtake not, been further ſhewn, that your Inferences are no more juſt than your Conceptions are clear, and that your Logics are as exceptionable as your Metaphyſics. It ſhould ſeem therefore upon the whole, that your Concluſions are not attained by juſt Reaſoning from clear Principles; conſequently that the Employment of modern Analyſts, however uſeful in Mathematical Calculations, and Conſtructions, doth not habituate and qualify the Mind to apprehend clearly and infer juſtly; and conſequently, that you have no Right in Virtue of ſuch Habits, to dictate out of your proper Sphere, beyond which your Judgment is to paſs for no more than that of other Men.

L. Of a long Time I have ſuſpected, that theſe modern Analytics were not ſcientifical, and gave ſome Hints thereof to the Publick about twenty five Years ago. Since which Time, I have been diverted by other Occupations, and imagined I might employ my ſelf better than in deducing and laying together my Thoughts on ſo nice a Subject. And though of late I have been called upon to make good my Suggeſtions; yet as the Perſon, who made this Call, doth not appear to think maturely enough to underſtand, either thoſe Metaphyſics which he would refute, or Mathematics which he would patronize, I ſhould have ſpared my ſelf the Trouble of Writing for his Conviction. Nor ſhould I now have troubled you or my ſelf with this Addreſs, after ſo long an Intermiſſion of theſe Studies; were it not to prevent, ſo far as I am able, your impoſing on your ſelf and others in Matters of much higher Moment and Concern. And to the End that you may more clearly comprehend the Force and Deſign of the foregoing Remarks, and purſue them ſtill further in your own Meditations, I ſhall ſubjoin the following Queries.

Query 1. Whether the Object of Geometry be not the Proportions of aſſignable Extenſions? And whether, there be any need of conſidering Quantities either infinitely great or infinitely ſmall?

Qu. 2. Whether the End of Geometry be not to meaſure aſſignable finite Extenſion? And whether this practical View did not firſt put Men on the ſtudy of Geometry?

Qu. 3. Whether the miſtaking the Object and End of Geometry hath not created needleſs Difficulties, and wrong Purſuits in that Science?

Qu. 4. Whether Men may properly be ſaid to proceed in a ſcientific Method, without clearly conceiving the Object they are converſant about, the End propoſed, and the Method by which it is purſued?

Qu. 5. Whether it doth not ſuffice, that every aſſignable Number of Parts may be contained in ſome aſſignable Magnitude? And whether it be not unneceſſary, as well as abſurd, to ſuppoſe that finite Extenſion is infinitely diviſible?

Qu. 6. Whether the Diagrams in a Geometrical Demonſtration are not to be conſidered, as Signs of all poſſible finite Figures, of all ſenſible and imaginable Extenſions or Magnitudes of the ſame Kind?

Qu. 7. Whether it be poſſible to free Geometry from inſuperable Difficulties and Abſurdities, ſo long as either the abſtract general Idea of Extenſion, or abſolute external Extenſion be ſuppoſed its true Object?

Qu. 8. Whether the Notions of abſolute Time, abſolute Place, and abſolute Motion be not moſt abſtractedly Metaphyſical? Whether it be poſſible for us to meaſure, compute, or know them?

Qu. 9. Whether Mathematicians do not engage themſelves in Diſputes and Paradoxes, concerning what they neither do nor can conceive? And whether the Doctrine of Forces be not a ſufficient Proof of this? *

Qu. 10. Whether in Geometry it may not ſuffice to conſider aſſignable finite Magnitude, without concerning our ſelves with Infinity? And whether it would not be righter to meaſure large Polygons having finite Sides, inſtead of Curves, than to ſuppoſe Curves are Polygons of infiniteſimal Sides, a Suppoſition neither true nor conceivable?

Qu. 11. Whether many Points, which are not readily aſſented to, are not nevertheleſs true? And whether thoſe in the two following Queries may not be of that Number.

Qu. 12. Whether it be poſſible, that we ſhould have had an Idea or Notion of Extenſion prior to Motion? Or whether if a Man had never perceived Motion, he would ever have known or conceived one Thing to be diſtant from another?

Qu. 13. Whether Geometrical Quantity hath coexiſtent Parts? And whether all Quantity be not in a flux as well as Time and Motion?

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* See a *Latin* Treatiſe *De Motu*, publiſhed at *London* in the Year 1721.

Qu. 14. Whether Extenſion can be ſuppoſed an Attribute of a Being immutable and eternal?

Qu. 15. Whether to decline examining the Principles, and unravelling the Methods uſed in Mathematicks, would not ſhew a bigotry in Mathematicians?

Qu. 16. Whether certain Maxims do not paſs current among Analyſts, which are ſhocking to good Senſe? And whether the Common Aſſumption that a finite Quantity divided by nothing is infinite be not of this Number?

Qu. 17. Whether the conſidering Geometrical Diagrams abſolutely or in themſelves, rather than as Repreſentatives of all aſſignable Magnitudes or Figures of the ſame Kind, be not a principal Cauſe of the ſuppoſing finite Extenſion infinitely diviſible; and of all the Difficulties and Abſurdities conſequent thereupon?

Qu. 18. Whether from Geometrical Propoſitions being general, and the Lines in Diagrams being therefore general Subſtitutes or Repreſentatives, it doth not follow that we may not limit or conſider the Number of Parts, into which ſuch particular Lines are diviſible?

Qu. 19. When it is ſaid or implied, that ſuch a certain Line delineated on Paper contains more than any aſſignable Number of Parts, whether any more in Truth ought to be underſtood, than that it is a Sign indifferently repreſenting all finite Lines, be they ever ſo Great. In which relative Capacity it contains, *i. e.* ſtands for more than any aſſignable Number of Parts? And whether it be not altogether abſurd to ſuppoſe a finite Line, conſidered in itſelf or in its own poſitive Nature, ſhould contain an infinite Number of Parts?

Qu. 20. Whether all Arguments for the infinite Diviſibility of finite Extenſion do not ſuppoſe and imply, either general abſtract Ideas or abſolute external Extenſion to be the Object of Geometry? And, therefore, whether, along with thoſe Suppoſitions, ſuch Arguments alſo do not ceaſe and vaniſh?

Qu. 21. Whether the ſuppoſed infinite Diviſibility of finite Extenſion hath not been a Snare to Mathematicians, and a Thorn in their Sides? And whether a Quantity infinitely diminiſhed and a Quantity infinitely ſmall are not the ſame Thing?

Qu. 22. Whether it be neceſſary to conſider Velocities of naſcent or evaneſcent Quantities, or Moments, or Infiniteſimals? And whether the introducing of Things ſo inconceivable be not a Reproach to Mathematicks?

Qu. 23. Whether Inconſiſtencies can be Truths? Whether Points repugnant and abſurd are to be admitted upon any Subject, or in any Science? And whether the Uſe of Infinites ought to be allowed, as a ſufficient Pretext and Apology, for the admitting of ſuch Points in Geometry?

Qu. 24. Whether a Quantity be not properly ſaid to be known, when we know its Proportion to given Quantities? And whether this Proportion can be known, but by Expreſſions or Exponents, either Geometrical, Algebraical, or Arithmetical? And whether Expreſſions in Lines or Species can be uſeful but ſo far forth as they are reducible to Numbers?

Qu. 25. Whether the finding out proper Expreſſions or Notations of Quantity be not the moſt general Character and Tendency of the Mathematicks? And Arithmetical Operation that which limits and defines their Uſe?

Qu. 26. Whether Mathematicians have ſufficiently conſidered the Analogy and Uſe of Signs? And how far the ſpecific limited Nature of Things correſponds thereto?

Qu. 27. Whether becauſe, in ſtating a general Caſe of pure Algebra, we are at full liberty to make a Character denote, either a poſitive or a negative Quantity, or nothing at all, we may thereſore in a Geometrical Caſe, limited by Hypotheſes and Reaſonings from particular Properties and Relations of Figures, claim the ſame Licenſe?

Qu. 28. Whether the Shifting of the Hypotheſis, or (as we may call it) the *fallacia Suppoſitionis* be not a Sophiſm, that far and wide infects the modern Reaſonings, both in the mechanical Philoſophy and in the abſtruſe and fine Geometry?

Qu. 29. Whether we can form an Idea or Notion of Velocity diſtinct from and excluſive of its Meaſures, as we can of Heat diſtinct from and excluſive of the Degrees on the Thermometer, by which it is meaſured? And whether this be not ſuppoſed in the Reaſonings of modern Analyſts?

Qu. 30. Whether Motion can be conceived in a Point of Space? And if Motion cannot, whether Velocity can? And if not, whether a firſt or laſt Velocity can be conceived in a mere Limit; either Initial or Final, of the deſcribed Space?

Qu. 31. Where there are no Increments, whether there can be any *Ratio* of Increments? Whether Nothings can be conſidered as proportional to real Quantities? Or whether to talk of their Proportions be not to talk Nonſenſe? Alſo in what Senſe we are to underſtand the Proportion of a Surface to a Line, of an Area to an Ordinate? And whether Species or Numbers, though properly expreſſing Quantities which are not homogeneous, may yet be ſaid to expreſs their Proportion to each other?

Qu. 32. Whether if all aſſignable Circles may be Squared, the Circle is not, to all Intents and Purpoſes, ſquared as well as the Parabola? Or whether a parabolical Area can in Fact be meaſured more accurately than a Circular?

Qu. 33. Whether it would not be righter to approximate fairly, than to endeavour at Accuracy by Sophiſms?

Qu. 34. Whether it would not be more decent to proceed by Tryals and Inductions, than to pretend to demonſtrate by falſe Principles?

Qu. 35. Whether there be not a Way of arriving at Truth, although the Principles are not ſcientific, nor the Reaſoning juſt? And whether ſuch a Way ought to be called a Knack or a Science?

Qu. 36. Whether there can be Science of the Concluſion, where there is not Evidence of the Principles? And whether a Man can have Evidence of the Principles, without underſtanding them? And therefore whether the Mathematicians of the preſent Age act like Men of Science, in taking ſo much more Pains to apply their Principles, than to underſtand them?

Qu. 37. Whether the greateſt Genius wreſtling with falſe Principles may not be foiled?

And whether accurate Quadratures can be obtained without new *Poſtulata* or Aſſumptions? And if not, whether thoſe which are intelligible and conſiſtent ought not to be preferred to the Contrary? *See* Sect. XXVIII. *and* XXIX.

Qu. 38. Whether tedious Calculations in Algebra and Fluxions be the likelieſt Method to improve the Mind? And whether Mens being accuſtomed to reaſon altogether about Mathematical Signs and Figures, doth not make them at a loſs how to reaſon without them?

Qu. 39. Whether, whatever Readineſs Analyſts acquire in ſtating a Problem, or finding apt Expreſſions for Mathematical Quantities, the ſame doth neceſſarily infer a proportionable Ability in conceiving and expreſſing other Matters?

Qu. 40. Whether it be not a general Caſe or Rule, that one and the ſame Coefficient dividing equal Products gives equal Quotients? And yet whether ſuch Coefficient can be interpreted by *o* or nothing? Or whether any one will ſay, that if the Equation $2 \times o = 5 \times o$, be divided by *o*, the Quotients on both Sides are equal? Whether therefore a Caſe may not be general with reſpect to all Quantities, and yet not extend to Nothings, or include the Caſe of Nothing? And whether the bringing Nothing under the Notion of Quantity may not have betrayed Men into falſe Reaſoning?

Qu. 41. Whether in the moſt general Reaſonings about Equalities and Proportions, Men may not demonſtrate as well as in Geometry? Whether in ſuch Demonſtrations, they are not obliged to the ſame ſtrict Reaſoning as in Geometry? And whether ſuch their Reaſonings are not deduced from the ſame Axioms with thoſe in Geometry? Whether therefore Algebra be not as truly a Science as Geometry.

Qu. 42. Whether Men may not reaſon in Species as well as in Words? Whether the ſame Rules of Logic do not obtain in both Caſes? And whether we have not a Right to expect and demand the ſame Evidence in both.

Qu. 43. Whether an Algebraiſt, Fluxioniſt, Geometrician or Demonſtrator of any kind can expect Indulgence for obſcure Principles or incorrect Reaſonings? And whether an Algebraical Note, or Species can at the he End of a Proceſs be interpreted in a Senſe, which could not have been ſubſtituted for it at the Beginning? Or whether any particular Suppoſition can come under a general Caſe which doth not conſiſt with the reaſoning thereof?

Qu. 44. Whether the Difference between a mere Computer and a Man of Science be not, that the One computes on Principles clearly conceived, and by Rules evidently demonſtrated, whereas the Other doth not?

Qu. 45. Whether, although Geometry be a Science, and Algebra allowed to be a Science, and the Analytical a moſt excellent Method, in the Application nevertheleſs of the Analyſis to Geometry, Men may not have admitted falſe Principles and wrong Methods of Reaſoning?

Qu. 46. Whether, although Algebraical Reaſonings are admitted to be ever ſo juſt, when confined to Signs or Species as general Repreſentatives of Quantity, you may not nevertheleſs fall into Error, if, when you limit them to ſtand for particular Things, you do not limit your ſelf to reaſon conſiſtently with the Nature of ſuch particular Things?

And whether ſuch Error ought to be imputed to pure Algebra?

Qu. 47. Whether the View of modern Mathematicians doth not rather ſeem to be the coming at an Expreſſion by Artifice, than the coming at Science by Demonſtration?

Qu. 48. Whether there may not be ſound Metaphyſics as well as unſound? Sound as well as unſound Logic? And whether the modern Analytics may not be brought under one of theſe Denominations, and which?

Qu. 49. Whether there be not really a *Philoſophia prima*, a certain tranſcendental Science ſuperior to and more extenſive than Mathematicks, which it might behove our modern Analyſts rather to learn than deſpiſe?

Qu. 50. Whether ever ſince the Recovery of Mathematical Learning, there have not been perpetual Diſputes and Controverſies among the Mathematicians? And whether this doth not diſparage the Evidence of their Methods?

Qu. 51. Whether any Thing but Metaphyſics and Logic can open the Eyes of Mathematicians and extricate them out of their Difficulties?

Qu. 52. Whether upon the received Principles a Quantity can by any Diviſion or Subdiviſion, though carried ever ſo far, be reduced to nothing?

Qu. 53. Whether if the End of Geometry be Practice, and this Practice be Meaſuring, and we meaſure only aſſignable Extenſions, it will not follow that unlimited Approximations completely anſwer the Intention of Geometry?

Qu. 54. Whether the ſame Things which are now done by Infinites may not be done by finite Quantities? And whether this would not be a great Relief to the Imaginations and Underſtandings of Mathematical Men?

Qu. 55. Whether thoſe Philomathematical Phyſicians, Anatomiſts, and Dealers in the Animal Oeconomy, who admit the Doctrine of Fluxions with an implicit Faith, can with a good Grace inſult other Men for believing what they do not comprehend?

Qu. 56. Whether the Corpuſcularian, Experimental, and Mathematical Philoſophy ſo much cultivated in the laſt Age, hath not too much engroſſed Mens Attention; ſome part whereof it might have uſefully employed?

Qu. 57. Whether from this, and other concurring Cauſes, the Minds of ſpeculative Men have not been born downward, to the debaſing and ſtupifying of the higher Faculties? And whether we may not hence account for that prevailing Narrowneſs and Bigotry among many who paſs for Men of Science, their Incapacity for Things Moral, Intellectual, or Theological, their Proneneſs to meaſure all Truths by Senſe and Experience of animal Life?

Qu. 58. Whether it be really an Effect of Thinking, that the ſame Men admire the great Author for his Fluxions, and deride him for his Religion?

Qu. 59. If certain Philoſophical Virtuoſi of the preſent Age have no Religion, whether it can be ſaid to be want of Faith?

Qu. 60. Whether it be not a juſter Way of Reaſoning, to recommend Points of Faith from their Effects, than to demonſtrate Mathematical Principles by their Concluſions?

Qu. 61. Whether it be not leſs exceptionable to admit Points above Reaſon than contrary to Reaſon?

Qu. 62. Whether Myſteries may not with better Right be allowed of in Divine Faith, than in Humane Science?

Qu. 63. Whether ſuch Mathematicians as cry out againſt Myſteries, have ever examined their own Principles?

Qu. 64. Whether Mathematicians, who are ſo delicate in religious Points, are ſtrictly ſcrupulous in their own Science? Whether they do not ſubmit to Authority, take things upon Truſt, believe Points inconceivable? Whether they have not their Myſteries, and what is more, their Repugnancies and Contradictions?

Q. 65. Whether it might not become Men, who are puzzled and perplexed about their own Principles, to judge warily, candidly, and modeſtly concerning other Matters?

Qu. 66. Whether the modern Analytics do not furniſh a ſtrong *argumentum ad hominem*, againſt the Philomathematical Infidels of theſe Times?

Qu. 67. Whether it follows from the abovementioned Remarks, that accurate and juſt Reaſoning is the peculiar Character of the preſent Age? And whether the modern Growth of Infidelity can be aſcribed to a Diſtinction ſo truly valuable?

FINIS.